Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 2, pp. 213–219, 2009

METHOD OF *A*-OPERATORS AND CONSERVATION LAWS FOR THE EQUATIONS OF GAS DYNAMICS

Yu. A. Chirkunov

UDC 517.944+533.6

An algorithm is proposed which allows all conservation laws for a system of differential equations to be to obtained from its one zero-order conservation law for which the general rank of the Jacobi matrix is equal to the number of independent variables of the system. The efficiency of the algorithm is shown by examples of the equations of gas dynamics, for which new conservation laws are derived. For the equations considered, additional symmetry properties related to these conservation laws are established.

Key words: *operator, conservation law, classification of the equations of irrotational gas motion, gas dynamics, symmetry, equivalence.*

INTRODUCTION

Conservation laws for various systems of differential equations have been studied in many papers. The authors of these papers find these laws by direct calculations based on the determination of the conservation law or by using generalized symmetries of the systems or the Nöther theorem. As a rule, the objects of research are systems of differential equations corresponding to various models of mechanics and physics complicated by additional effects. Studies of the conservation laws for the equations of gas dynamics have yielded the following results. Using direct calculations, Shmyglevskii [1] derived the complete system of zero-order conservation laws for the equations of motion for an perfect gas in the three-dimensional case. Applying the point symmetry operators to the classical conservation laws, Ibragimov [2] obtained additional conservation laws for the equations of irrotational motion of a polytropic gas.

In the present study, the n-dimensional ($n \geqslant 1$) equations of irrotational gas motion are classified according to the zero-order conservation laws using the $\boldsymbol{A}$-operator method proposed by the author (all generalized symmetries of the system of differential equations are a subset of the set of its $\boldsymbol{A}$-operators). A new equation of state (Chaplygin generalized gas) is obtained for which there is an extension of the set of conservation laws.

1. METHOD OF *A*-OPERATORS

We consider an arbitrary system (S) of differential equations for m ($m \geqslant 1$) unknown functions $\boldsymbol{u} = (u^1, u^2, \ldots, u^m)$ of $n+1$ ($n \geqslant 1$) independent variables $\boldsymbol{y} = (x^0, x^1, x^2, \ldots, x^n)$. Let $[S]$ be a manifold in extended space defined by the equations of system (S) and all its differential consequences. The conservation law for system (S) is a vector $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{y}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}, \underset{2}{\boldsymbol{u}}, \ldots) = (A^0, A^1, A^2, \ldots, A^n)$ such that $(\boldsymbol{D} \cdot \boldsymbol{A})_{[S]} = 0$, where

Novosibirsk State University of Economics and Management, Novosibirsk 630070; chr01@rambler.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 2, pp. 53–60, March–April, 2009. Original article submitted May 22, 2008; revision submitted October 6, 2008.

$\boldsymbol{D} = (D_0, D_1, D_2, \ldots, D_n)$; $D_i = D_{x^i}$ is the operator of total differentiation with respect to the variable x^i $(i = 0, 1, 2, \ldots, n)$; $\underset{k}{\boldsymbol{u}}$ $(k = 1, 2, \ldots)$ is the set of functions $D_{i_1} D_{i_2} \cdots D_{i_k} \boldsymbol{u}$ $(i_m = 0, \ldots, n;\ m = 1, \ldots, k)$ [3].

Definition 1. Let $\boldsymbol{A}$ be a conservation law for system (S). Then, by virtue of system (S) and all its differential consequences, the evolutionary generalized-symmetry operator $X = \boldsymbol{\eta}(\boldsymbol{y}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}, \underset{2}{\boldsymbol{u}}, \ldots) \cdot \partial_{\boldsymbol{u}} + \ldots$ admitted by the equation $\boldsymbol{D} \cdot \boldsymbol{A} = 0$ is called an $\boldsymbol{A}$-operator of this system:

$$(X(\boldsymbol{D} \cdot \boldsymbol{A}))_{[S]} = 0. \tag{1}$$

For the set of $\boldsymbol{A}$-operators of system (S), the lower-bound estimate can be indicated: this set contains the Lie algebra of all generalized symmetries of system (S). The relationship between the $\boldsymbol{A}$-operators of system and the conservation laws of this system leads to the following two propositions.

Proposition 1. The action of any $\boldsymbol{A}$-operator of system (S) on the conservation law $\boldsymbol{A}$ gives a conservation law for this system.

Proposition 2. Let $\boldsymbol{A}$ be a conservation law of system (S). Then, any evolutionary generalized-symmetry operator $X = \boldsymbol{\eta}(\boldsymbol{y}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}, \underset{2}{\boldsymbol{u}}, \ldots) \cdot \partial_{\boldsymbol{u}} + \ldots$ for which the vector $X\boldsymbol{A}$ is a conservation law of this system is its $\boldsymbol{A}$-operator.

Definition 2. The $\boldsymbol{A}$-operator X of system (S) is called its trivial $\boldsymbol{A}$-operator if the vector $X\boldsymbol{A}$ is a trivial conservation law for this system.

Definition 3. Two $\boldsymbol{A}$-operators of system (S) are called $\boldsymbol{A}$-equivalent if their difference is a trivial $\boldsymbol{A}$-operator of this system.

Application of the $\boldsymbol{A}$-equivalent $\boldsymbol{A}$-operators of system (S) to the conservation law $\boldsymbol{A}$ yields equivalent conservation laws for this system. Hence, the set of $\boldsymbol{A}$-operators of system (S) for each conservation law $\boldsymbol{A}$ is divided into classes of $\boldsymbol{A}$-equivalent $\boldsymbol{A}$-operators.

The following theorem of the generating conservation law for systems of differential equations holds.

Theorem 1. *If the system of differential equations (S) has a zero-order conservation law $\boldsymbol{A}$ for which the total rank of the Jacobi matrix $\partial \boldsymbol{A}/\partial \boldsymbol{u}$ is equal to the number of independent variables of system (S), each conservation law for this system can be obtained by applying some $\boldsymbol{A}$-operator of this system to the conservation law $\boldsymbol{A}$.*

The method of obtaining conservation laws for systems of differential equations by means of theorem 1 will be called the $\boldsymbol{A}$-operator method.

2. CONSERVATION LAWS FOR THE EQUATIONS OF GAS DYNAMICS

The equations of motion for a gas with the calorific the equation of state $p = f(\rho, S)$ (S is the entropy) are written as

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{1}{\rho} \nabla p = \boldsymbol{0}, \qquad \rho_t + (\boldsymbol{u} \cdot \nabla)\rho + \rho \operatorname{div} \boldsymbol{u} = 0,$$
$$p_t + (\boldsymbol{u} \cdot \nabla)p + \rho c^2(p, \rho) \operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

where t is time, $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is the velocity vector, $\rho = \rho(t, \boldsymbol{x})$ is the density, $p = p(t, \boldsymbol{x})$ is the pressure, $\boldsymbol{x} \in \mathbb{R}^n$ $(n \geqslant 1)$, and $c = c(p, \rho) > 0$ is the sound velocity.

Irrotational gas motion is described by Eqs. (2) and the equations

$$\nabla \rho \wedge \nabla p = \boldsymbol{0}, \qquad \nabla \wedge \boldsymbol{u} = \boldsymbol{0}, \tag{3}$$

where the $\wedge$ symbol denotes the external multiplication operator. By introducing the potential $\varphi = \varphi(t, \boldsymbol{x})$, we integrate the second equation in (3):

$$\boldsymbol{u} = \nabla \varphi. \tag{4}$$

In gas dynamics, the physical meaning of the conservation law $\boldsymbol{A} = (A^0, A^1, A^2, \ldots, A^n)$ is determined by the component A^0 — the density of the conservation law and the flux vector $\boldsymbol{B} = \boldsymbol{A}_1 - A^0 \boldsymbol{u}$, where $\boldsymbol{A}_1 = (A^1, A^2, \ldots, A^n)$.

2.1. Conservation Laws for the Equations of One-Dimensional Gas Motion. The generating conservation law $\boldsymbol{A}$ for Eq. (2) for $n = 1$ is taken to be the conservation law defining the motion of the center of mass:

$$A^0 = \rho(tu - x), \qquad B = tp. \tag{5}$$

From the system of constitutive equations obtained from relation (1), it follows that the quotient set of zero-order $\boldsymbol{A}$-operators for system (2) for $n = 1$ for the set of its trivial $\boldsymbol{A}$-operators in the case of an arbitrary function $c^2(p, \rho)$ is generated by the operators

$$\begin{gathered} X_1 = \frac{p}{(tu-x)\rho}\,\partial_u + \frac{(tu-x)\rho u - tp}{(tu-x)^2}\,\partial_\rho, \\ X_2 = \frac{pu}{(tu-x)\rho}\,\partial_u + \frac{\rho(tu-x)(u^2/2+\varepsilon) - tpu}{(tu-x)^2}\,\partial_\rho, \\ X_3 = \frac{\rho g(S)}{tu-x}\,\partial_\rho, \qquad X_4 = \frac{tp}{(tu-x)\rho}\,\partial_u + \frac{\rho(tu-x)^2 - t^2 p}{(tu-x)^2}\,\partial_\rho, \end{gathered} \tag{6}$$

where $\varepsilon = \varepsilon(p, \rho)$ is the specific internal energy and $g = g(S)$ is an arbitrary function of the entropy S.

Extension of the set of nontrivial zero-order $\boldsymbol{A}$-operators for system (2) ($n = 1$) occurs in only two cases: 1) for $c^2 = 3p/\rho$, i.e., for a polytropic gas with an adiabatic exponent equal to 3; 2) for $c^2 = \theta(p)/\rho^2$]$\theta(p)$ is an arbitrary given function], i.e., for a gas which will be called the generalized Chaplygin gas [for $\theta(p) = \mathrm{const}$ this is the well-known Chaplygin gas].

If $c^2 = 3p/\rho$, operators (6) are supplemented by the $\boldsymbol{A}$-operators

$$\begin{gathered} X_5 = \frac{(2tu-x)p}{(tu-x)\rho}\,\partial_u + \frac{t^2 u(\rho u^2 - p) - x\rho u(2tu-x)}{(tu-x)^2}\,\partial_\rho, \\ X_6 = \frac{2tp}{\rho}\,\partial_u + \frac{t^2(\rho u^2 - p) - x\rho(2tu-x)}{(tu-x)^2}\,\partial_\rho. \end{gathered} \tag{7}$$

If $c^2 = \theta(p)/\rho^2$, then, for $n = 1$, the set of nontrivial $\boldsymbol{A}$-operators for system (2) consists of the operators X_3 and X_4 and an infinite family of operators

$$X_7 = \frac{\theta H_p}{(tu-x)\rho}\,\partial_u + \frac{\rho(tu-x)H_u - t\theta H_p}{(tu-x)^2}\,\partial_\rho, \tag{8}$$

where $H = H(u, p)$ is any solution of the equation

$$H_{uu} = [\theta(p)H_p]_p. \tag{9}$$

If $c^2(p, \rho)$ is an arbitrary function, then, for $n = 1$, by applying the operators X_1, X_2, X_3, and X_4 to the conservation law (5), we obtain the following classical conservation laws for system (2): the laws of conservation of momentum, energy, and entropy [for $g(S) \equiv \mathrm{const}$, the mass conservation law] and the conservation law (5) defining the motion of the center of mass [the operator X_4 acts on the conservation law (5) identically].

If $c^2 = 3p/\rho$, application of the operators X_5 and X_6 to the conservation law (5) yields the additional two conservation laws found in [2].

If $c^2 = \theta(p)/\rho^2$, then by applying the operators X_3, X_4, and X_7 to the conservation law (5), we obtain all classical conservation laws and an infinite family of conservation laws

$$A^0 = \rho H_u, \qquad B = \theta(p)H_p, \tag{10}$$

where $H = H(u, p)$ is a solution of Eq. (9).

Thus, we proved the following theorem.

Theorem 2. *If $c^2(p, \rho)$ is an arbitrary function, then, for $n = 1$, the set of nontrivial zero-order conservation laws for system* (2) *includes only the classical conservation laws. Extension of this set takes place in only two cases:* 1) *for $c^2 = 3p/\rho$, where the conservation laws obtained in* [2] *are added;* 2) *for $c^2 = \theta(p)/\rho^2$* [*$\theta(p)$ is a given function*], *where the set of nontrivial zero-order conservation laws for system* (2) *for $n = 1$ includes the classical conservation laws and an infinite set of conservation laws* (10).

A group classification of system (2) was performed in [3]. For $n = 1$, the case $c^2 = 3p/\rho$ corresponds to the widest basic group of Lie transformations of this system, and the case $c^2 = \theta(p)/\rho^2$ is not distinguished in the group classification. However, $c^2 = \theta(p)/\rho^2$ is the single value of the coefficient c^2 for which system (2) has generalized symmetries of the first order. Direct calculations lead to the following statement.

Theorem 3. *For $n = 1$, system (2) has generalized symmetries of the first order which are not equivalent to point symmetries if and only if $c^2 = \theta(p)/\rho^2$ [$\theta(p)$ is a given function]. If $c^2 = \theta(p)/\rho^2$, the set of first-order operators admitted by this system which are not equivalent to the point-symmetry operators whose coordinates depend linearly on the derivatives u_x, p_x, and ρ_x consists of the operators*

$$X_g = gu_x\,\partial_u + gp_x\,\partial_p + (\rho g)_x\,\partial_\rho + \ldots; \tag{11}$$

$$X_h = \Big[\Big(\frac{\theta}{\rho}\,h_p + h\Big)u_x + \frac{h_u}{\rho}\,p_x\Big]\partial_u + \Big(\frac{\theta}{\rho}\,h_x + hp_x\Big)\partial_p + (\rho h)_x\,\partial_\rho + \ldots, \tag{12}$$

where $g = g(S)$ is an arbitrary function of the entropy S and $h = h(u, p)$ is an arbitrary solution of Eq. (9).

Operator (11) is a first-order $\boldsymbol{A}$-operator for system (2) ($n = 1$) for all $c^2(p, \rho)$. The result of the action of this operator on the conservation law (5) is the entropy conservation law (to within a first-order trivial conservation law).

Operator (12) is a first-order $\boldsymbol{A}$-operator for system (2) ($n = 1$) for all $c^2 = \theta(p)/\rho^2$. Applying these operators with the function $h = H_u(u, p)$ [H is a solution of Eq. (9)] to the conservation law (5), we obtain the conservation law (10) (to within a first-order trivial conservation law).

2.2. Conservation Laws for the Equations of Irrotational Gas Motion. Irrotational gas motion is described by system (2), (3) ($n \geqslant 2$). The generating conservation law $\boldsymbol{A}$ for this system is taken to be the momentum conservation law

$$A^0 = \rho\boldsymbol{u}\cdot\boldsymbol{b}, \qquad \boldsymbol{B} = p\boldsymbol{b}, \tag{13}$$

where $\boldsymbol{b}$ is a fixed constant unit vector.

The solution of the system of constitutive equations obtained from relation (1) shows that, for an arbitrary function $c^2(p, \rho)$, the set of nontrivial zero-order $\boldsymbol{A}$-operators of system (2), (3) is defined by the operators

$$Y_1 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{p}{\rho}\,\boldsymbol{u}\cdot\partial_{\boldsymbol{u}} + \Big(\rho\Big(\frac{|\boldsymbol{u}|^2}{2} + \varepsilon\Big) - p\Big)\partial_\rho\Big], \qquad Y_2 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{p}{\rho}\,\boldsymbol{l}\cdot\partial_{\boldsymbol{u}} - \Big(\rho\boldsymbol{u} - \frac{p\boldsymbol{b}}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big)\cdot\boldsymbol{l}\,\partial_\rho\Big],$$

$$Y_3 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{p}{\rho}\,Q\langle\boldsymbol{x}\rangle\cdot\partial_{\boldsymbol{u}} + Q\langle\boldsymbol{x}\rangle\cdot\Big(\rho\boldsymbol{u} - \frac{p\boldsymbol{b}}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big)\partial_\rho\Big], \tag{14}$$

$$Y_4 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{tp}{\rho}\,\boldsymbol{k}\cdot\partial_{\boldsymbol{u}} + \Big(\rho(t\boldsymbol{u} - \boldsymbol{x}) - \frac{tp\boldsymbol{b}}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big)\cdot\boldsymbol{k}\,\partial_\rho\Big], \qquad Y_5 = \frac{\rho g(S)}{\boldsymbol{u}\cdot\boldsymbol{b}}\,\partial_\rho,$$

where Q is an arbitrary antisymmetric tensor of rank 2 in $\mathbb{R}^n$, $\boldsymbol{l}$ and $\boldsymbol{k}$ are any constant unit vectors, $g = g(S)$ is an arbitrary function of the entropy S and $\varepsilon = \varepsilon(p, \rho)$ specific internal energy. Extension of this set occurs in only two cases: 1) for $c^2(p, \rho) = ((n + 2)/n)p/\rho$, where (14) is supplemented by $\boldsymbol{A}$-operators

$$Y_6 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{p}{\rho}\,(2t\boldsymbol{u} - \boldsymbol{x})\cdot\partial_{\boldsymbol{u}} + q\,\partial_\rho\Big],$$

$$Y_7 = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\Big[\frac{2tp}{\rho}\,(t\boldsymbol{u} - \boldsymbol{x})\cdot\partial_{\boldsymbol{u}} + (t^2(\rho|\boldsymbol{u}|^2 + (n - 2)p) + 2tq + \rho|\boldsymbol{x}|^2)\,\partial_\rho\Big] \tag{15}$$

$$\Big(q = p\,\frac{\boldsymbol{x}\cdot\boldsymbol{b}}{\boldsymbol{u}\cdot\boldsymbol{b}} + t(\rho|\boldsymbol{u}|^2 + (n - 2)p) - \rho(\boldsymbol{x}\cdot\boldsymbol{u})\Big);$$

2) for $c^2 = 1/(\rho^2 w''(p))$ [$w(p)$ is a given function for which $w''(p) \neq 0$], where (14) are supplemented by the $\boldsymbol{A}$-operators

$$Y_8 = \frac{1}{\rho(\boldsymbol{u}\cdot\boldsymbol{b})}\,[(\boldsymbol{u}\cdot\boldsymbol{\lambda})\boldsymbol{u} - W\boldsymbol{\lambda}]\cdot\partial_{\boldsymbol{u}} + (\boldsymbol{z}\cdot\boldsymbol{\lambda})\,\partial_\rho,$$

$$Y_9 = \frac{1}{\rho(\boldsymbol{u}\cdot\boldsymbol{b})}\,[(\Omega\langle\boldsymbol{x}\rangle\cdot\boldsymbol{u})\boldsymbol{u} - W\Omega\langle\boldsymbol{x}\rangle]\cdot\partial_{\boldsymbol{u}} + (\Omega\langle\boldsymbol{x}\rangle\cdot\boldsymbol{z})\partial_\rho,$$

$$Y_{10} = \frac{1}{\rho(\boldsymbol{u}\cdot\boldsymbol{b})}\Big[((t\boldsymbol{u}-\boldsymbol{x})\cdot\boldsymbol{\mu})\boldsymbol{u} + \frac{\boldsymbol{u}\cdot\boldsymbol{\mu}}{n-1}\,\boldsymbol{x} - \Big(tW + \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{n-1}\Big)\boldsymbol{\mu}\Big]\cdot\partial_{\boldsymbol{u}} \tag{16}$$

$$+\Big[(t\boldsymbol{z} - (\rho w'(p)-1)(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{x})\cdot\boldsymbol{\mu} + \frac{1}{n-1}\,((\boldsymbol{\mu}\cdot\boldsymbol{b})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{b})\boldsymbol{\mu})\cdot\boldsymbol{u}\Big]\partial_\rho$$

$$\Big(W = \frac{|\boldsymbol{u}|^2}{2} - w(p), \qquad \boldsymbol{z} = \frac{1}{\boldsymbol{u}\cdot\boldsymbol{b}}\,[(\boldsymbol{u}\cdot\boldsymbol{b})(\rho w'(p)-1)\boldsymbol{u} + W\boldsymbol{b}]\Big),$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are arbitrary constant vectors and Ω is an arbitrary antisymmetric tensor of rank 2 in $\mathbb{R}^n$.

By applying the operators Y_1, Y_2, Y_3, Y_4, and Y_5 to the conservation law (13) for system (2), (3), we obtain the following conservation laws, respectively:

— the energy conservation law

$$A^0 = \rho(|\boldsymbol{u}|^2/2 + \varepsilon), \qquad \boldsymbol{B} = p\boldsymbol{u}; \tag{17}$$

— the momentum conservation law

$$A^0 = \rho\boldsymbol{u}\cdot\boldsymbol{l}, \qquad \boldsymbol{B} = p\boldsymbol{l}; \tag{18}$$

— the angular momentum conservation law

$$A^0 = \rho Q\langle\boldsymbol{x}\rangle\cdot\boldsymbol{u}, \qquad \boldsymbol{B} = pQ\langle\boldsymbol{x}\rangle; \tag{19}$$

— the conservation law defining the motion of the center of mass:

$$A^0 = \rho(t\boldsymbol{u}-\boldsymbol{x})\cdot\boldsymbol{k}, \qquad \boldsymbol{B} = tp\cdot\boldsymbol{k}; \tag{20}$$

— the entropy conservation law [for $g(S) \equiv$ const the mass conservation law]

$$A^0 = \rho g(S), \qquad \boldsymbol{B} = \boldsymbol{0}. \tag{21}$$

For $c^2(p,\rho) = ((n+2)/n)p/\rho$, applying the operators Y_6 and Y_7 to the conservation law (13), we have the additional two conservation laws found in [2]:

$$A^0 = t(\rho|\boldsymbol{u}|^2 + np) - \rho\boldsymbol{x}\cdot\boldsymbol{u}, \qquad \boldsymbol{B} = p(2t\boldsymbol{u}-\boldsymbol{x}); \tag{22}$$

$$A^0 = t^2(\rho|\boldsymbol{u}|^2 + np) - \rho\boldsymbol{x}\cdot(2t\boldsymbol{u}-\boldsymbol{x}), \qquad \boldsymbol{B} = 2tp(t\boldsymbol{u}-\boldsymbol{x}). \tag{23}$$

For $c^2 = 1/(\rho^2 w''(p))$, by applying the operators Y_8, Y_9, and Y_{10} to the conservation law (13), we obtain three additional conservation laws, respectively:

— the additional generalized momentum conservation law

$$A^0 = \rho w'(p)\boldsymbol{u}\cdot\boldsymbol{\lambda}, \qquad \boldsymbol{B} = [w(p) - |\boldsymbol{u}|^2/2]\boldsymbol{\lambda} + (\boldsymbol{u}\cdot\boldsymbol{\lambda})\boldsymbol{u}; \tag{24}$$

— the additional generalized angular momentum conservation law

$$A^0 = \rho w'(p)Q\langle\boldsymbol{x}\rangle\cdot\boldsymbol{u}, \qquad \boldsymbol{B} = [w(p) - |\boldsymbol{u}|^2/2]Q\langle\boldsymbol{x}\rangle + (Q\langle\boldsymbol{x}\rangle\cdot\boldsymbol{u})\boldsymbol{u}; \tag{25}$$

— the law of conservation defining the additional generalized law of motion of the center of mass:

$$A^0 = \rho w'(p)(t\boldsymbol{u}-\boldsymbol{x})\cdot\boldsymbol{\mu},$$

$$\boldsymbol{B} = \Big[t\Big(w(p) - \frac{|\boldsymbol{u}|^2}{2}\Big) - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{n-1}\Big]\boldsymbol{\mu} + [(t\boldsymbol{u}-\boldsymbol{x})\cdot\boldsymbol{\mu}]\boldsymbol{u} + \frac{\boldsymbol{u}\cdot\boldsymbol{\mu}}{n-1}\,\boldsymbol{x}. \tag{26}$$

Thus, we proved the following theorem.

Theorem 4. *The set of nontrivial zero-order conservation laws for system* (2), (3) *describing irrotational gas motion for an arbitrary function* $c^2(p,\rho)$ *consists of the classical conservation laws* (13) *and* (17)–(21). *Extesion of this set occurs in only two cases:* 1) *for* $c^2 = ((n+2)/n)p/\rho$, *where the conservation laws* (22) *and* (23) *are added;* 2) *for* $c^2 = 1/(\rho^2 w''(p))$ [$w(p)$ *is a given function such that* $w''(p) \neq 0$], *where the conservation laws* (24)–(26) *are added.*

2.3. Nonlocal Conservation Laws for the Equations of Irrotational Gas Motion. For Eqs. (2), the potential $\varphi = \varphi(t, \boldsymbol{x})$ is a nonlocal variable. For system (2)–(4), conservation laws $\boldsymbol{A}$ are sought in the form

$$\boldsymbol{A} = \boldsymbol{A}(t, \boldsymbol{x}, \boldsymbol{u}, p, \rho, \varphi, \varphi_t). \tag{27}$$

One-Dimensional Case. For $n = 1$, as the generating conservation law $\boldsymbol{A}$ for system (2), (4), we use the conservation law defining the law of motion of the center of mass (5). The $\boldsymbol{A}$-operator for this system is sought in the form

$$\eta\, \partial_u + \sigma\, \partial_p + \tau\, \partial_\rho + \psi\, \partial_\varphi + \dots, \tag{28}$$

where η, σ, τ, and ψ are unknown functions of the variables t, x, u, p, ρ, φ, and φ_t.

From the system of constitutive equations obtained from (1), it follows that for an arbitrary function $c^2(p, \rho)$, the set of nontrivial $\boldsymbol{A}$-operators of the form (28) for system (2), (4) consists of operators (6). Extension of this set occurs in only two cases: 1) for $c^2 = 3p/\rho$, where operators (7) are added; 2) for $c^2 = 1/(\rho^2 w''(p))$ [$w(p)$ is a given function such that $w''(p) \neq 0$] where the set of nontrivial $\boldsymbol{A}$-operators of the form (28) for system (2), (4) consists of operators (6) and (8) for which $\theta(p) = 1/w''(p)$ and the operator

$$X_8 = [\rho w'(p) + 1]\, \partial_\rho + [w(p) - u^2/2 - \varphi_t]\, \partial_p + \dots\ . \tag{29}$$

By applying operator (29) to the conservation law (5), for system (2), (4) we obtain the conservation law

$$A^0 = \rho w'(p)(tu - x), \qquad B = t[w(p) + u^2/2] - xu + \varphi, \tag{30}$$

which is a nonlocal conservation law with a nonlocal variable φ for Eqs. (2).

The results obtained can be formulated as the following theorem.

Theorem 5. *For $n = 1$, system* (2) *has nontrivial nonlocal conservation laws of the form* (27) *if and only if $c^2 = 1/(\rho^2 w''(p))$. Here $w(p)$ is a given function for which $w''(p) \neq 0$* (*generalized Chaplygin gas*). *Each of these conservation law is equivalent to the conservation law* (30).

Multidimensional Case. For $n \geqslant 2$, the generating conservation law $\boldsymbol{A}$ for system (2)–(4) is taken to be the momentum conservation law (13). For this system, the $\boldsymbol{A}$-operator is sought in the form

$$\boldsymbol{\eta} \cdot \partial_{\boldsymbol{u}} + \sigma\, \partial_p + \tau\, \partial_\rho + \psi\, \partial_\varphi + \dots\ , \tag{31}$$

where $\boldsymbol{\eta}$, σ, τ, and ψ are unknown functions of the variables t, $\boldsymbol{x}$, $\boldsymbol{u}$, p, ρ, φ, and φ_t.

The set of nontrivial $\boldsymbol{A}$-operators of the form (31) for system (2)–(4) with an arbitrary function $c^2(p, \rho)$ is generated by operators (14). Extension of this set occurs in only two cases: 1) for $c^2(p, \rho) = ((n+2)/n)p/\rho$, where operators (15) are added; 2) for $c^2 = 1/(\rho^2 w''(p))$ [$w(p)$ is a given function for which $w''(p) \neq 0$], where operators (16) and the following operator [an n-dimensional analog of operator (29)] are added:

$$Y_{11} = [\rho w'(p) + 1]\, \partial_\rho + [w(p) - |\boldsymbol{u}|^2/2 - \varphi_t]\, \partial_p + \dots\ . \tag{32}$$

System (2) does not have other nonlocal $\boldsymbol{A}$-operators of the form (31) that are not $\boldsymbol{A}$-equivalent to the operator (32). By applying the operator Y_{11} to the conservation law (13), we obtain a conservation law which is equivalent to the conservation law (24) and is independent of φ and φ_t. Consequently, in the n-dimensional case ($n \geqslant 2$), system (2) does not have nononlocal conservation laws of the form (27).

It turns out that Y_{11} is a nontrivial $\boldsymbol{A}$-operator not only for the conservation law (13) but also for each of the conservation laws (18)–(20). By applying the operator Y_{11} to conservation laws (19) and (20), we obtain conservation laws equivalent to the conservation laws (25), (26).

As shown by direct calculations, the group classification of system (2)–(4) for the coefficient $c^2(p, \rho)$ according to the operators admitted by this system

$$\xi^O\, \partial_t + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} + \boldsymbol{\eta} \cdot \partial_{\boldsymbol{u}} + \sigma\, \partial_p + \tau\, \partial_\rho + \psi\, \partial_\psi + \dots\ , \tag{33}$$

where ξ^0, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, σ, τ, and ψ are unknown functions of the variables t, $\boldsymbol{x}$, $\boldsymbol{u}$, p, ρ, φ, and φ_t coincides with known group classification of system (2) according to the point operators admitted by this system [the corresponding Lie algebras are only extended to the operator $f(t)\, \partial_\varphi$ with an arbitrary function $f(t)$] [3]. Furthermore, the case $c^2 = 1/(\rho^2 w''(p))$ is not isolated. However, if one searches for operators of the form (33) admitted, by virtue of (2)–(4), by the system consisting of all equations of system (2) except for the last equation, then, to within obvious equivalence transformations, only two cases occur: 1) $c^2(p, \rho)$ is an arbitrary function; 2) $c^2 = 1/(\rho^2 w''(p))$ [$w(p)$ is a given function for which $w''(p) \neq 0)$]. If $c^2(p, \rho$ is an arbitrary function, the set of the indicated operators has the form

$$\partial_t, \quad \partial_{\boldsymbol{x}}, \quad \boldsymbol{x} \wedge \partial_{\boldsymbol{x}} + \boldsymbol{u} \wedge \partial_{\boldsymbol{u}}, \quad t\,\partial_{\boldsymbol{x}} + \partial_{\boldsymbol{u}} + \boldsymbol{x}\,\partial_\varphi, \quad \alpha(t)\,\partial_\varphi, \quad \beta(t)\,\partial_p,$$

$$t\,\partial_t + \boldsymbol{x} \cdot \partial_{\boldsymbol{x}} + \varphi\,\partial_\varphi, \quad t\,\partial_t - \boldsymbol{u} \cdot \partial_{\boldsymbol{u}} + 2\rho\,\partial_\rho - \varphi\,\partial_\varphi, \quad \rho\,\partial_\rho + p\,\partial_p, \tag{34}$$

$$t^2\,\partial_t + t\boldsymbol{x} \cdot \partial_{\boldsymbol{x}} + (\boldsymbol{x} - t\boldsymbol{u}) \cdot \partial_{\boldsymbol{u}} - nt\rho\,\partial_\rho - (n+2)tp\,\partial_p + (1/2)|\boldsymbol{x}|^2\,\partial_\varphi,$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions. If $c^2 = 1/(\rho^2 w''(p))$, operators (34) are supplemented by the nonlocal operator (32).

It is necessary to note that set (34), to within the operators $\alpha(t)\,\partial_\varphi$ and $\beta(t)\,\partial_p$ [$\beta'(t) \neq 0$], consists of all point operators admitted by system (2), in particular, for all cases of extension of its basic algebra.

This work was supported by the Russian Foundation for Basic Research (Grant No. 07-01-00489).

REFERENCES

1. Yu. D. Shmyglevskii, *Analytical Investigations of Fluid Dynamics* [in Russian], Éditorial URSS, Moscow (1999).
2. N. Kh. Ibragimov, *Transformation Groups in Mathematical Physics* [in Russian], Nauka, Moscow (1983).
3. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).